



SMALL-TIME ANALYTICAL SOLUTION OF PHASE CHANGE PROBLEMS IN CYLINDRICAL AND SPHERICAL DOMAINS

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Abstract: Numerical methods are usually employed to solve heat transfer problems with phase change in cylindrical and spherical domains since there is no analytical solution to such problems. Since the problems are singular at initial time, this singularity must be eliminated by the use of a starting small-time solution. In this work, phase change problems with Dirichlet boundary conditions in cylindrical and spherical domains have been solved analytically using Landau's Transform around the initial time. It has been shown that the small-time solution is identical to Neumann's solution.

Key Words: Phase Change; Neumann's Solution; Stefan Problem

FAZ DEĞİŞİMLİ ISI İLETİM PROBLEMLERİNİN SİLİNDİRİK VE KÜRESEL UZAYDA BAŞLANGIÇ ANI İÇİN ANALİTİK ÇÖZÜMÜ

Özet: Silindirik ve küresel uzayda faz değişimli ısı iletimi problemlerinin analitik çözümleri olmadığından, çözüm için sayısal yöntemler kullanılır. Faz değişim problemleri başlangıç anında tekillik içerdiklerinden, sayısal çözüme başlamak için küçük-zaman çözümlerine gerek duyulur. Bu çalışmada, silindirik ve küresel uzayda Dirichlet sınır koşullarına sahip faz değişim problemleri, başlangıç anı için analitik olarak çözülmüş ve çözümün Neuman çözümüne özdeş olduğu gösterilmiştir.

Anahtar kelimeler: Faz Değişimi; Neumann Çözümü; Stefan Problemi

NOMENCLATURE

C_p	specific heat, $\text{Jkg}^{-1}\text{K}^{-1}$
k	thermal conductivity, $\text{Wm}^{-1}\text{K}^{-1}$
k^*	the ratio of liquid thermal conductivity to solid thermal conductivity
L	latent heat, Jkg^{-1}
r	radial coordinate, m
R	dimensionless radial coordinate
s	interface location, m
S	dimensionless interface location
Ste	$=C_p(T_m - T_o)/L$, Stefan number
t	time, s
T	temperature, K
α	thermal diffusivity, m^2s^{-1}
α^*	the ratio of solid thermal diffusivity to liquid thermal diffusivity
β	fixed coordinate
γ	1 or 2, for cylindrical and spherical domains, respectively
λ	the root of transcendental equation (40)
ρ	density, kgm^{-3}
τ	dimensionless time
θ	dimensionless temperature

Subscripts

i	initial
m	melting
o	cylinder surface
l	liquid
s	solid

INTRODUCTION

Heat exchange problems including solidification or melting processes are called heat conduction problems with phase change. The problems involving unknown boundaries are inherently non-linear even for linear differential equations. The location of interface is a part of the problem in addition to the temperature distribution in the solid and liquid phases. If the temperature gradient appears in only one phase then the problem is called one-phase otherwise two-phase (Özışık, 1993). It can be encountered with the phase change problems in many applications such as ice production, freezing of foods, casting, and latent heat thermal energy storage. Furthermore, one can face to phase change problems in designing of buildings and pipe lines in cold climates (Lunardini, 1981).

There are few exact solutions about phase change problems for only some idealized situations subject to simple boundary and initial conditions

(Carslaw and Jaeger, 1959). For the situations for which the exact solutions are not available; approximate, semi-analytical and numerical methods have been used to solve the phase change problems. An extensive review of these methods can be found in Crank (1984).

The problem has singularity at the initial time. This singularity can be eliminated by the use of a starting small-time solution. Small-time solution can be performed by assuming temperature profiles satisfying boundary conditions such as linear, parabolic and logarithmic profiles. The profiles are introduced into interface velocity equation and the location of interface is obtained. If the assumed temperature profiles do not fit the nature of the problem, the solution of the problem is not accurate enough. Therefore, it is important that the small-time solution closes to exact solution with increasing time.

The aim of this work is to find out the exact small-time solution that is employed outward phase change problems with first kind boundary conditions in cylindrical and spherical domains. For this purpose, applying the front-fixing transformation which is first proposed by Landau (1950) for heat diffusion equation and limiting process $\tau \rightarrow 0$ to solid and liquid phase equations and interface velocity equation, two quasi-steady differential equations have been found. The differential equations have been solved analytically and then exact solution for small times was obtained. The result showed that exact small-time solution is identical to Neumann's solution.

PROBLEM DESCRIPTION

Consider a liquid at a uniform temperature T_i , which is higher than the melting temperature T_m of the solid phase around a cylinder or sphere with radii r_o . When the surface temperature of cylinder or sphere is reduced to a temperature of T_o , which is less than T_m suddenly, the solidification will start. If the physical properties are independent of temperature and no density change occurs during solidification process, the mathematical formulation of this problem is given as

$$\frac{\partial^2 T_s}{\partial r^2} + \frac{\gamma}{r} \frac{\partial T_s}{\partial r} = \frac{1}{\alpha_s} \frac{\partial T_s}{\partial t} \quad r_o < r < s(t), t > 0 \quad (1)$$

$$\frac{\partial^2 T_l}{\partial t^2} + \frac{\gamma}{r} \frac{\partial T_l}{\partial r} = \frac{1}{\alpha_l} \frac{\partial T_l}{\partial t} \quad s(t) < r, t > 0 \quad (2)$$

where $s(t)$ is the location of the interface and γ equals to unity and 2 in cylindrical and spherical domains, respectively. The initial and boundary conditions of the problem are given by

$$T_s(r_o, t) = T_o \quad (3)$$

$$T_s(s(t), t) = T_m \quad (4)$$

$$T_l(s(t), t) = T_m \quad (5)$$

$$T_l(r \rightarrow \infty, t) \rightarrow T_i \quad (6)$$

$$T_l(r, 0) = T_i \quad (7)$$

$$\left[k_s \frac{\partial T_s}{\partial r} - k_l \frac{\partial T_l}{\partial r} \right]_{r=s(t)} = \rho_s L \frac{ds}{dt} \quad (8)$$

and

$$s(0) = r_o \quad (9)$$

T_o , T_m and T_i in the above equations are the surface temperature of the cylinder or the sphere, the interface temperature and the initial temperature, respectively, and L is the latent heat of solidification. Equation (8) is named as 'interface velocity equation' in the literature.

Using the dimensionless parameters

$$R = \frac{r}{r_o} \quad S = \frac{s(t)}{r_o} \quad \tau = \frac{\alpha_s t}{r_o^2} \quad (10)$$

$$\alpha^* = \frac{\alpha_s}{\alpha_l} \quad k^* = \frac{k_l}{k_s} \quad Ste = \frac{C_{p,s}(T_m - T_o)}{L}$$

$$\theta_s = \frac{T_s - T_o}{T_m - T_o} \quad \theta_l = \frac{T_l - T_o}{T_m - T_o}$$

the formulation of the problem reduces to

$$\frac{\partial^2 \theta_s}{\partial R^2} + \frac{\gamma}{R} \frac{\partial \theta_s}{\partial R} = \frac{\partial \theta_s}{\partial \tau} \quad (11)$$

$$\frac{\partial^2 \theta_l}{\partial R^2} + \frac{\gamma}{R} \frac{\partial \theta_l}{\partial R} = \frac{1}{\alpha^*} \frac{\partial \theta_s}{\partial \tau} \quad (12)$$

$$\theta_s(R = 1, \tau) = 0 \quad (13)$$

$$\theta_s(R = S, \tau) = 1 \quad (14)$$

$$\theta_l(R = S, \tau) = 1 \quad (15)$$

$$\theta_l(R \rightarrow \infty, \tau) \rightarrow \theta_i \quad (16)$$

$$\theta_l(R, 0) = \theta_i \quad (17)$$

$$\left[\frac{\partial \theta_s}{\partial R} - k^* \frac{\partial \theta_l}{\partial R} \right]_{R=S} = \frac{1}{Ste} \frac{\partial S}{\partial \tau} \quad (18)$$

$$S(0) = 1 \quad (19)$$

Using the front-fixing transformation that was first proposed by Landau (1950), the interface is fixed. Thus, movement of the interface is inserted into the solid and liquid phase equations and interface velocity equation. New independent variable is chosen as,

$$\beta = \frac{R-1}{S-1} \quad (20)$$

Dimensionless energy equation for cylindrical coordinates can be expressed as follows.

$$\frac{D\theta}{D\tau} = \frac{\partial \theta}{\partial \tau} + V_r \frac{\partial \theta}{\partial R} + \frac{V_\phi}{R} \frac{\partial \theta}{\partial \phi} + V_z \frac{\partial \theta}{\partial z} \quad (21)$$

where θ , τ , V_r , V_ϕ , V_z dimensionless temperature, dimensionless time and velocities of the coordinates, respectively.

Because β coordinate in the phase change problem which is examined in this work is a function of interface location, it has a velocity. Therefore, the first term at right-hand side of Equation (21) should be substituted into governing equations in the form of Equation (22).

$$\frac{\partial \theta}{\partial \tau} \rightarrow \frac{\partial \theta}{\partial \tau} + \frac{\partial \theta}{\partial \beta} \frac{\partial \beta}{\partial \tau} \quad (22)$$

Taking derivative of β with respect to dimensionless time gives the following;

$$\frac{\partial \theta}{\partial \tau} \rightarrow \frac{\partial \theta}{\partial \tau} + \frac{\partial \theta}{\partial \beta} \left(-\frac{dS}{d\tau} \frac{\beta}{S-1} \right) \quad (23)$$

If Equation (23) is inserted into Equations (11) and (12), the governing equations and their initial and boundary conditions can be written as Equations (24)-(32).

$$\frac{\partial \theta_s}{\partial \tau} = \frac{1}{(S-1)^2} \frac{\partial^2 \theta_s}{\partial \beta^2} + \left[\frac{\gamma}{\beta(S-1)+1} \frac{1}{S-1} + \frac{\beta}{S-1} \frac{dS}{d\tau} \right] \frac{\partial \theta_s}{\partial \beta} \quad (24)$$

$$\frac{\partial \theta_l}{\partial \tau} = \frac{1}{\alpha^* (S-1)^2} \frac{\partial^2 \theta_l}{\partial \beta^2} + \left[\frac{\gamma}{\alpha^* \beta(S-1)+1} \frac{1}{S-1} + \frac{\beta}{S-1} \frac{dS}{d\tau} \right] \frac{\partial \theta_l}{\partial \beta} \quad (25)$$

$$\theta_s(0, \tau) = 0 \quad (26)$$

$$\theta_s(1, \tau) = 1 \quad (27)$$

$$\theta_l(1, \tau) = 1 \quad (28)$$

$$\theta_l(\beta \rightarrow \infty, \tau) = \theta_i \quad (29)$$

$$\theta_l(\beta, 0) = \theta_i \quad (30)$$

$$\frac{1}{(S-1)} \left[\frac{\partial \theta_s}{\partial \beta} - k^* \frac{\partial \theta_l}{\partial \beta} \right]_{\beta=1} = \frac{1}{Ste} \frac{dS}{d\tau} \quad (31)$$

$$S(0) = 1 \quad (32)$$

SMALL-TIME ANALYTICAL SOLUTION

The term which contains the temperature gradients in the interface velocity equation (31) equals to a constant.

$$Ste \left[\frac{\partial \theta_s}{\partial \beta} - k^* \frac{\partial \theta_l}{\partial \beta} \right]_{\beta=1} = C^2 \quad (33)$$

The interface velocity equation (31) can be rewritten as

$$\frac{dS}{d\tau} = \frac{C^2}{S-1} \quad (34)$$

When Equation (34) is solved analytically, Equation (35) is obtained.

$$S = 1 + C\sqrt{2\tau} \quad (35)$$

Substituting Equations (34) and (35) into solid and liquid phase equations, which are expressed in Equations (24) and (25), and then multiplying Equation (24) by $2C^2\tau$ and (25) by $2\alpha^*C^2\tau$, in the limit $\tau \rightarrow 0$, solid and liquid phase equations take the forms

$$\frac{d^2 \theta_s}{d\beta^2} + C^2 \beta \frac{d\theta_s}{d\beta} = 0, \quad 0 < \beta < 1 \quad (36)$$

$$\frac{d^2 \theta_l}{d\beta^2} + \alpha^* C^2 \beta \frac{d\theta_l}{d\beta} = 0, \quad \beta > 1 \quad (37)$$

Assigning $C^2 = 2\lambda^2$ and applying the boundary conditions to Equations (36) and (37), we then obtain

$$\theta_s(\beta) = \frac{\text{erf}(\lambda\beta)}{\text{erf}(\lambda)} \quad (38)$$

$$\theta_l(\beta) = 1 + (\theta_i - 1) \left[1 - \frac{\text{erfc}(\sqrt{\alpha^*} \lambda \beta)}{\text{erfc}(\sqrt{\alpha^*} \lambda)} \right] \quad (39)$$

Substituting Equations (38) and (39) into Equation (33), it is obtained the transcendental equation which gives the value of λ .

$$\frac{\exp(-\lambda^2)}{\text{erf}(\lambda)} - k^* \sqrt{\alpha^*} (\theta_i - 1) \frac{\exp(-\alpha^* \lambda^2)}{\text{erfc}(\sqrt{\alpha^*} \lambda)} = \frac{\lambda \sqrt{\pi}}{Ste} \quad (40)$$

Using inverse front-fixing transformation and letting $x=R-1$, the analytical solution which is known as Neumann's Solution in Cartesian coordinates is evaluated.

DISCUSSION

The small-time analytical solution obtained for spherical and cylindrical outward phase change problems in this work was found identical to Neumann's solution that is exact solution of the problem with the same boundary and initial conditions in Cartesian coordinates. The small-time solution is valid for cylindrical and spherical domains around the initial time.

The constant temperature boundary conditioned one-phase problem was solved by the use of Integral Method by Lardner and Pohle (1961), Sparrow et al. (1978), Bell (1979),

Lunardini (1981), Voller and Cross (1981) and Ilken (1987).

Assuming the dimensionless temperature profile is linear, Ilken (1987) solved one-phase problem by the using integral method and obtained the interface velocity equation as,

$$\frac{dS}{d\tau} = \left(\frac{1 - \ln(S)}{S - 1} \right) \left(\frac{2Ste}{2 + Ste} \right) \quad (41)$$

The solution is singular for $\tau=0$ at which time $S=1$. This singularity was eliminated by the use of a starting small-time solution. The small-time solution that was first proposed by Poots (1962) is given as following equation.

$$S = 1 + e_0\tau^{1/2} + e_1\tau + e_2\tau^{3/2} + \dots \quad (42)$$

Substituting Equation (42) into Equation (41), Ilken (1987) calculated the coefficients of terms having same order in each side of Equation (41). Since the solution obtained in this study is an exact solution, there is no need to guess a small-time solution anymore. The small-time solutions obtained in this work are compared with those of Ilken's (1987) in Table 1. The differences between present small-time solution and Ilken's are getting larger with increasing Ste number as shown in Table 1.

Table 1. Location of the interface for different Ste Number

τ	Ste	Ilken (1987)	Present study	Difference %
10^{-6}	0.1	1.00043668772448	1.00044003254120	3.340000E-04
	1.0	1.00115536739604	1.00124012505852	8.465268E-03
	10	1.00182679688616	1.00251394390743	6.854239E-02
	100	1.00198143961406	1.00370189523230	1.714110E-01
10^{-5}	0.1	1.00138264978769	1.00139150512223	8.843029E-04
	1.0	1.00365815646455	1.00392161990211	2.624343E-02
	10	1.00578407325839	1.00794978892850	2.148634E-01
	100	1.00627371430155	1.01170642099263	5.369845E-01
10^{-4}	0.1	1.00436739253382	1.00440032600227	3.278919E-03
	1.0	1.01161386435301	1.01240125044417	7.777411E-02
	10	1.01836376516622	1.02513944244672	6.609518E-01
	100	1.01991848970994	1.03701895190210	1.649002E-00
10^{-3}	0.1	1.01405218520626	1.01391505028579	1.352529E-02
	1.0	1.03718758916158	1.03921619638176	1.952055E-01
	10	1.05881876698756	1.07949788393462	1.915624E-00
	100	1.06380391181579	1.11706420204760	4.767881E-00
10^{-2}	0.1	1.04612848581598	1.04400323319279	2.035676E-01
	1.0	1.12232916896011	1.12401244687436	1.497562E-01
	10	1.19405266975039	1.25139427118566	4.582217E-00
	100	1.21066945847563	1.37018934717674	1.164218E-01

CONCLUSIONS

The following conclusions can be derived from the present work.

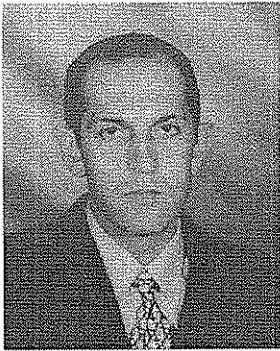
1. The present small-time solution is identical to Neumann's solution.

2. The solution is valid for both melting and solidification problems in cylindrical and spherical domains.

3. When a small-time solution is necessary for solving a phase change problem, one should use the exact small-time solution obtained in this work instead of any approximate solution such as Ilken's solution.

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