



## OPTIMUM DESIGN OF A RECTANGULAR FIN WITH TEMPERATURE-DEPENDENT THERMAL PARAMETERS

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**Abstract:** In the present work, rectangular fins with constant base temperature, non-symmetric heat transfer coefficients and temperature-dependent thermal conductivity have been considered. In order to transform non-linear two-dimensional heat diffusion equation into a linear equation, the Kirchhoff transformation has been used. Resulting linear partial differential equation has been solved analytically by using the separation of variables method. The optimum geometry which maximizes the heat transfer rate has been found employing NCONF routine in the IMSL Library.

**Key Words:** Rectangular fin, Extended surface, Optimization

## ISIL PARAMETRELERİ SICAKLIKLA DEĞİŞEN DİKDÖRTGEN KESİTLİ DÜZ BİR KANADIN OPTİMUM TASARIMI

**Özet:** Bu çalışmada, dip sıcaklığı sabit olan, simetrik olmayan ısı taşınım katsayılarına sahip ve ısı iletim katsayısı sıcaklıkla değişen dikdörtgen kesitli bir kanat göz önüne alınmıştır. Doğrusal olmayan iki boyutlu ısı yayılım denklemini doğrusal bir denkleme dönüştürmek için, Kirchhoff dönüşümü kullanılmıştır. Elde edilen doğrusal kısmi diferansiyel denklem, değişkenlerin ayrımı yöntemi ile analitik olarak çözülmüştür. Kanattan çevreye olan ısı geçişini maksimum yapan optimum kanat geometrisi, doğrusal olmayan optimizasyon problemini çözen IMSL NCONF FORTRAN kodu kullanılarak bulunmuştur.

**Anahtar Kelimeler:** Dikdörtgen kesitli düz kanat, Genişletilmiş yüzeyler, Optimizasyon

### NOMENCLATURE

|                |                                                        |
|----------------|--------------------------------------------------------|
| a              | fin length (m)                                         |
| b              | fin thickness (m)                                      |
| Bi             | Biot number                                            |
| h              | heat transfer coefficient [W/(m <sup>2</sup> K)]       |
| h <sub>0</sub> | a coefficient in Eq. 8 [W/(m <sup>2</sup> K)]          |
| k              | thermal conductivity [W/(mK)]                          |
| k <sub>0</sub> | thermal conductivity at reference temperature [W/(mK)] |
| N              | norms for y-direction                                  |
| Q              | heat transfer rate for unit depth (W/m)                |

|                      |                                  |
|----------------------|----------------------------------|
| q                    | dimensionless heat-transfer rate |
| T <sub>0</sub>       | reference temperature (K)        |
| T <sub>b</sub>       | base temperature (K)             |
| T <sub>∞</sub>       | ambient fluid temperature (K)    |
| U                    | new-dependent-variable in Eq. 4  |
| x                    | horizontal coordinate (m)        |
| X                    | eigenfunction in ξ-direction     |
| y                    | vertical coordinate (m)          |
| Y                    | eigenfunction in η-direction     |
| <b>Greek letters</b> |                                  |
| β <sub>m</sub>       | m. eigenvalue for η-direction    |
| δ                    | the fin ratio                    |

|           |                             |
|-----------|-----------------------------|
| $\eta$    | dimensionless coordinate    |
| $\xi$     | dimensionless coordinate    |
| $\lambda$ | a constant defined in Eq. 6 |
| $\Theta$  | dimensionless temperature   |

#### Subscripts and Superscripts

|     |                     |
|-----|---------------------|
| 1,2 | surfaces of the fin |
| 3   | tip of the fin      |
| (L) | lower               |
| (U) | upper               |
| *   | optimum             |

## INTRODUCTION

Finned surfaces have been in use for a long period as a heat dissipation mechanism. An extensive review is seen on this topic (Kern and Krause, 1972). Technology has led to a demand for high-performance, light-weight and compact heat transfer components. In order to accommodate the demand, finned surfaces are used to increase the heat transfer rate between a primary surface and the surrounding fluid in heat exchanging devices. Thus, optimization of the design of fins is of significant importance. The optimization of fins is generally based on two approaches: one is to minimize the volume or mass for a given amount of heat dissipation and the other is to maximize the heat dissipation for a given volume or mass (Lau and Tan, 1973; Mikk, 1980; Ullmann and Kalman, 1989). For purely conductive and convective fins, the criterion for the optimal problem was first proposed by Schmidt (1926).

Several papers are devoted to finding the shape of fins that would minimize the volume for a given amount of heat dissipation (Yeh and Liaw, 1993; Razani and Zohoor, 1991; Sohrabpour and Razani, 1993). However, all of these studies consider negligible heat transfer from the tip, because their fins have sharp tips. Apparently, the sharp-tip fin design has the disadvantage that the resulting profiles are too difficult to manufacture. Several other studies conducted to find the dimensions of a constant thickness straight fin that would maximize the heat dissipation for a given volume (Soon and Bar-Cohen, 1981; Li, 1983; Yeh, 1995a). These studies, however, invoked the assumption of insulated tips, and furthermore, they used one-dimensional heat transfer approach. It has been shown that the one-dimensional approach is convenient, but may be in error for certain physical conditions (Lau and Tan, 1973; Look and Kang, 1992; Yeh, 1995b; Look and Kang, 1991). With regard to the applicability of the one dimensional model for heat conduction in the fins, some arguments based either on the fin effectiveness (Bejan, 1993) or on evaluation of the largest

transverse variation of fin temperature (Snider and Kraus, 1983) are available to demonstrate that the one dimensional assumption is valid only for a transverse Biot number much less than unity,  $Bi \ll 1$ , a statement usually known as the 'validity criterion' (Casarosa and Franco, 2001). Aziz and Lunardini (1995) concluded that error on the dimensionless heat flow between the predictions of the one dimensional model and the correct two dimensional model, for a longitudinal rectangular fin, is approximately of 1% if  $Bi=0.1$ .

In this paper, optimization of rectangular fins with non-symmetric boundary conditions and variable thermal conductivity are considered. Resulting governing equation are transformed into a linear differential equation by using the Kirchhoff transformation. The problem is to maximize the heat-transfer rate for a given fin volume. The optimization variable is the ratio of fin height to fin length that will be called the fin ratio in the following sections. It is assumed that the predominant modes of heat transfer are conduction and convection, and the effect of radiation is ignored.

## PROBLEM DESCRIPTION

Consider a rectangular fin of a homogeneous material is attached to a base plate with a base temperature of  $T_b$ , which is measured in excess to the ambient fluid temperature,  $T_\infty$ . The geometry of the fin is shown schematically in Fig. 1. Considering steady-state conditions and also assuming that the thermal conductivity as a function of temperature, the radiation from the surfaces is negligible, the heat generation effects are absent, and the convection heat transfer coefficients  $h_1$ ,  $h_2$  and  $h_3$  are functions of temperature distribution on the surfaces; the equation for steady-state temperature distribution of the fin can be written as (Ozisik, 1980),

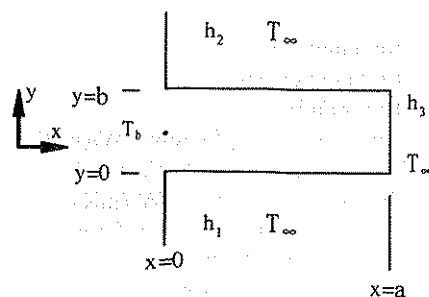


Fig. 1. Geometry of a two-dimensional rectangular fin

$$\frac{\partial}{\partial x} \left[ k(T) \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k(T) \frac{\partial T}{\partial y} \right] = 0 \quad (1a)$$

$$T = T_b \quad \text{for} \quad x = 0 \quad (1b)$$

$$k(T) \frac{\partial T}{\partial x} + h_3(T)(T - T_\infty) = 0 \quad \text{for} \quad x = a \quad (1c)$$

$$-k(T) \frac{\partial T}{\partial y} + h_1(T)(T - T_\infty) = 0 \quad \text{for} \quad y = 0 \quad (1d)$$

$$k(T) \frac{\partial T}{\partial y} + h_2(T)(T - T_\infty) = 0 \quad \text{for} \quad y = b \quad (1e)$$

The thermal conductivity of the fin material is assumed to be a linear function of temperature according to

$$k(T) = k_0 [1 + \gamma(T - T_0)] \quad (2)$$

where  $k_0$  is thermal conductivity of fin material at reference temperature,  $T_0$ , and  $\gamma$  is the dimensional parameter describing the variation of thermal conductivity.

Employing the following dimensionless parameters,

$$\Theta = \frac{T - T_0}{T_b - T_0} \quad \lambda = \gamma(T_b - T_0) \quad \xi = \frac{x}{a} \quad \eta = \frac{y}{a} \quad (3)$$

dimensionless governing equation is written as follows

$$\frac{\partial}{\partial \xi} \left[ (1 + \lambda \Theta) \frac{\partial \Theta}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[ (1 + \lambda \Theta) \frac{\partial \Theta}{\partial \eta} \right] = 0 \quad (4)$$

In order to transform Eq. (4) into a linear differential equation, the Kirchhoff transformation can be used (Ozisik, 1980). The dimensionless temperature  $\Theta$  is replaced in it by a new variable  $U = U(\Theta)$  such that

$$U(\Theta) = \int_{\Theta_0}^{\Theta} \frac{k(\Theta')}{k_0} d\Theta' \quad (5)$$

By using Eq.(5) in Eq.(4) we obtain

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = 0 \quad (6)$$

Taking the ambient fluid temperature as reference temperature, i.e.,  $\Theta_0 = 0$ , the transformation of boundary conditions is carried out as explained in the Appendix.

The boundary condition given in Eq. (1b) is transformed into the following condition.

$$U = 1 + \lambda/2 \quad \text{for} \quad \xi = 0 \quad (7)$$

The transformation of the other boundary conditions to linear boundary conditions of the third kind using the transformation is possible only under certain restrictions on the heat transfer coefficient  $h(\Theta)$  as mentioned in the Appendix. The restriction becomes the following expression according to Eq. (A.10).

$$h = h_0 \left( 1 + \frac{\lambda}{2} \Theta \right) \quad (8)$$

Unknown coefficient  $h_0$  can be calculated from known average heat transfer coefficient over the surface.

Defining two new dimensionless parameters named Biot number and the fin ratio, respectively, as

$$Bi = \frac{h_0 a}{k_0} \quad \delta = \frac{b}{a} \quad (9)$$

three linear boundary conditions are expressed as follows.

$$\frac{\partial U}{\partial \xi} + Bi_3 U = 0 \quad \text{for} \quad \xi = 1 \quad (10)$$

$$-\frac{\partial U}{\partial \eta} + Bi_1 U = 0 \quad \text{for} \quad \eta = 0 \quad (11)$$

$$\frac{\partial U}{\partial \eta} + Bi_2 U = 0 \quad \text{for} \quad \eta = \delta \quad (12)$$

## ANALYTICAL SOLUTION

An application of the method of separation of variables using:

$$U(\xi, \eta) = X(\xi)Y(\eta) \quad (13)$$

and using the three homogeneous boundary conditions, the new dependent variable is obtained by

$$U(\xi, \eta) = \sum_{m=1}^{\infty} C_m X(\beta_m, \xi) Y(\beta_m, \eta) \quad (14)$$

The eigenfunctions in Eq. (14) are expressed as

$$X(\beta_m, \xi) = \frac{(\beta_m + Bi_3)e^{\beta_m(1-\xi)} + (\beta_m - Bi_3)e^{-\beta_m(1-\xi)}}{(\beta_m + Bi_3)e^{\beta_m} + (\beta_m - Bi_3)e^{-\beta_m}}$$

and

$$Y(\beta_m, \eta) = \beta_m \cos(\beta_m \eta) + Bi_1 \sin(\beta_m \eta) \quad (15)$$

To determine the coefficients  $C_m$  in Eq. (14) we utilize non-homogeneous boundary condition given in Eq. (7) and the orthogonality property of the eigenfunction for  $\eta$  direction.

$$C_m = \frac{(1 + \lambda/2) \int_0^{\delta} Y(\beta_m, \eta') d\eta'}{\int_0^{\delta} [Y(\beta_m, \eta')]^2 d\eta'} \quad (16)$$

The term in the denominator of Eq. (16) is known as the norm, which can be integrated to obtain

$$N(\beta_m) = 2 \left[ (\beta_m^2 + Bi_1^2) \left( \delta + \frac{Bi_2}{\beta_m^2 + Bi_2^2} \right) + Bi_1 \right] \quad (17)$$

where the eigenvalues  $\beta_m$ 's are the positive roots of the following transcendental equation which is known as eigencondition

$$\tan(\beta_m \delta) = \frac{\beta_m (Bi_1 + Bi_2)}{\beta_m^2 - Bi_1 Bi_2} \quad (18)$$

The inverse Kirchhoff transformation from  $U$  to  $\Theta$  becomes

$$\Theta(\xi, \eta) = \frac{1}{\lambda} \left[ \sqrt{1 + 2\lambda U(\xi, \eta)} - 1 \right] \quad (19)$$

The dimensionless heat transfer rate is defined by applying the Fourier's law at the fin base, as follows

$$q = \frac{Q}{k_0 (T_b - T_{\infty})} = - \int_0^{\delta} \left( \frac{\partial \Theta}{\partial \xi} \right)_{\xi=0} d\eta \quad (20)$$

## OPTIMIZATION PROCEDURE

Many engineering problems can be formulated as a general nonlinear optimization problem. The statement of such a problem can be written as (Reklaitis et. al, 1983)

$$\text{Minimize } \phi(z) \quad z \in \mathcal{R}^N \quad (21a)$$

$$\text{Subject to } f_k(z) = 0 \quad k = 1, \dots, K \quad (21b)$$

$$g_j(z) \leq 0 \quad j = 1, \dots, J \quad (21c)$$

and

$$z_i^{(L)} \leq z_i \leq z_i^{(U)} \quad i = 1, \dots, N \quad (21d)$$

where,  $f_k(z)$  are equality constraints,  $g_j(z)$  are inequality constraints, and  $x_j^{(L)}$ ,  $x_j^{(U)}$  are the bounds on variables  $\{z\}$ .

The fin problem discussed in the present paper fits the definition of the nonlinear optimization problems mentioned above. The objective here is to maximize the dimensionless heat-transfer rate  $q$  (or minimize  $-q$ ). The optimization variable is the fin ratio  $\delta$ .

The dimensionless heat transfer rate, i.e., the objective function, is rewritten as follows

$$q = \int_0^{\delta} \left[ (1 + 2\lambda U)^{-1/2} \frac{\partial U}{\partial \xi} \right]_{\xi=0} d\eta \quad (22)$$

The optimization procedure uses the subroutine NCONF of IMSL library (1997). The routine NCONF is based on the subroutine NLPQL, a FORTRAN code developed by Schittkowski (1986). It uses a successive quadratic programming method to solve the general nonlinear programming problem.

## RESULTS AND DISCUSSION

The present analysis is valid for rectangular fins with thermally non-symmetric boundary conditions and temperature-dependent thermal conductivity. Using this analysis, the thermal performance of a rectangular fin has been investigated for variation of Biot numbers. The eigenvalues necessary for obtaining the heat transfer rate have been evaluated using an

efficient method given by Haji-Sheikh and Beck (2000) instead of classical Newton-Raphson procedure. Thus, an anticipated missing eigenvalue is avoided.

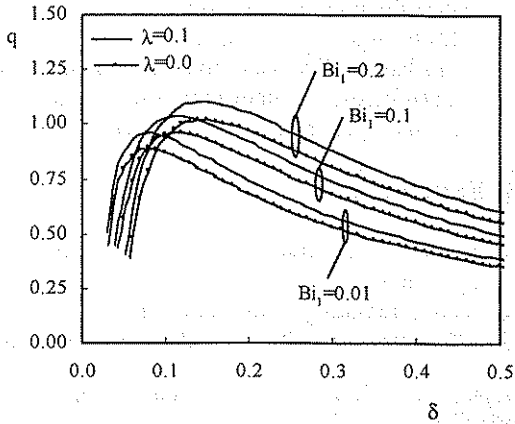


Fig. 2. The heat transfer rate as a function of the fin ratio ( $Bi_2=0.2$ ,  $Bi_3=0.0$ )

Table 1. Optimum values in Fig.2

| $Bi_1$ | $\lambda=0.0$ |        | $\lambda=0.1$ |        |
|--------|---------------|--------|---------------|--------|
|        | $\delta^*$    | $q^*$  | $\delta^*$    | $q^*$  |
| 0.01   | 0.079         | 0.9598 | 0.080         | 0.8907 |
| 0.10   | 0.113         | 1.0363 | 0.113         | 0.9596 |
| 0.20   | 0.144         | 1.1021 | 0.144         | 1.0193 |

Fig. 2 illustrates the heat transfer rate as a function of the fin ratio,  $\delta$ , and two different values of  $\lambda$  which describes the variation of the thermal conductivity. That  $\lambda$  equals zero corresponds the case of the constant thermal conductivity. Because  $B_3$  equals zero, this case corresponds to the assumption that the convective heat loss from the fin tip is negligible. The fin ratio decreases as the cross-sectional area of the fin decreases. This decrease in the cross-sectional area yields two effects to the heat transfer rate. The former, because of the increasing heat convection surfaces, the heat transfer rates increases and the latter, the heat conduction from the fin base decreases. The latter plays a dominant role for values of the fin ratio that is greater than a certain value. From Fig. 2, it can be seen that  $\delta^*$  corresponding the maximum heat transfer rate depends on Biot numbers. According to this result, the maximum heat transfer rate is displaced to the greater fin ratios with increasing Biot number on the surface 1. Values of the

optimum fin ratios and heat transfer rates in Fig. 2 have been shown in Table 1.

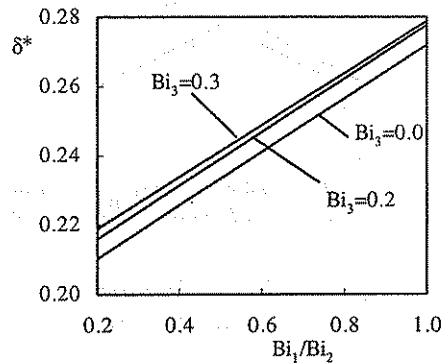


Fig.3. Optimum fin ratio for constant thermal parameters ( $Bi_2=0.5$ )

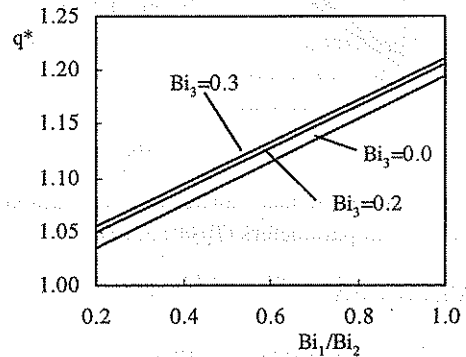


Fig.4. Maximum heat transfer rate for constant thermal parameters, ( $Bi_2=0.5$ )

Fig. 3 and Fig. 4 present the optimum fin ratio and the heat transfer rate as a function of  $B_1/B_2$ , for the case of constant thermal parameters, respectively. Increasing Biot number at the fin tip, the optimum fin ratio increases. Fig. 5 shows the optimum fin ratio, as a function of  $B_1/B_2$  for three different values of  $B_3$  and  $\lambda=0.1$ . Fig. 6 shows maximum values of the heat transfer rate corresponding to the optimum fin ratios which are given in Fig. 5. The heat transfer coefficients over the surfaces decrease with the temperature of the fin surfaces for  $\lambda>0$ . Therefore the heat transfer rate is less than that of the case of constant thermal properties.

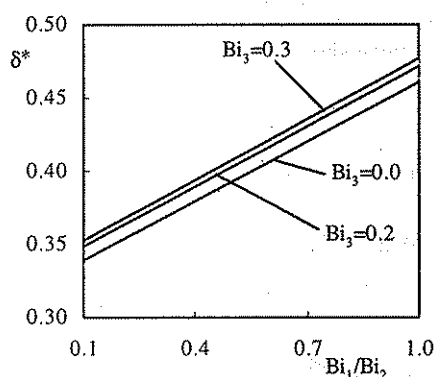


Fig.5. Optimum fin ratio for variable thermal parameters ( $Bi_2=0.5$ ,  $\lambda=0.1$ )

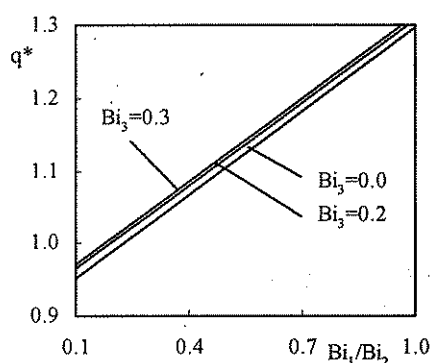


Fig.6. Maximum heat transfer rate for variable thermal parameters ( $Bi_2=0.5$ ,  $\lambda=0.1$ )

## CONCLUSIONS

Two-dimensional thermal analysis and optimization of rectangular fins with thermally non-symmetric boundary conditions and temperature-dependent thermal conductivity have been discussed in the present paper. The temperature distribution and the heat transfer rate have been obtained analytically. The optimum design of the rectangular fins has been obtained using NCONF routine, which uses successive quadratic programming method, in the FORTRAN IMSL library. Optimum fin ratios and maximum heat transfer rates for various combinations of Biot numbers have been presented in this work to provide an open choice to the designer.

The following concluding remarks can also be derived:

1. There always exists an optimum fin ratio of a rectangular fin which maximizes heat transfer rate.

2. The fin ratio of an optimum fin decreases with increasing heat transfer coefficients over the fin surfaces and at the fin tip.
3. The thermal analysis in the present paper can also be employed to analyze the fins with variable base temperatures.
4. Since the total heat transfer rate given in Eq. (22) is an analytical expression, optimum fin ratio has readily been obtained for different values of the problem parameters which are not employed in the present paper.

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## APPENDIX

### Kirchhoff Transformation of Boundary Condition of the First Kind (Ozisik, 1980)

Let the temperature at the boundary be prescribed as

$$T = f_i(\mathbf{r}, t) \quad \text{on } S_i \quad (\text{A.1})$$

The transformation of this boundary condition by the transformation Eq.(5) is also a boundary condition of the first kind. To illustrate this, consider the thermal conductivity to depend on temperature in the form

$$k(T) = k_0(1 + \lambda T) \quad \text{on } S_i \quad (\text{A.2})$$

Then, the transformation Eq. (5) becomes

$$U = \int_{T_0}^T (1 + \lambda T') dT' \quad (\text{A.3})$$

and the transformation of the boundary condition (A.1) gives

$$U = (f_i - T_0) + \frac{\lambda}{2} (f_i^2 - T_0^2) \quad (\text{A.4})$$

If the reference temperature  $\Theta_0$  is taken as zero, the transformation equation (A.4) reduces to

$$U = f_i + \frac{\lambda}{2} f_i^2 \quad (\text{A.5})$$

### Kirchhoff Transformation of Boundary Condition of the Third Kind (Ozisik, 1980)

The boundary condition is of the third kind in the form

$$k(\Theta) \frac{\partial \Theta}{\partial n_i} + h_i(\Theta) \cdot \Theta = f_i(\mathbf{r}, t) \quad \text{on } S_i \quad (\text{A.6})$$

The transformation of this boundary condition to a linear boundary condition of the third kind, using the transformation Eq. (5), is possible only under certain restrictions on the heat transfer coefficient  $h(\Theta)$  as illustrated below.

Suppose we wish to transform (A.6) into a linear boundary condition in the variable  $U$  as

$$k_0 \frac{\partial U}{\partial n_i} + h_{0,i} U = f_i(\mathbf{r}, t) \quad \text{on } S_i \quad (\text{A.7})$$

where  $k_0$  and  $h_{0,i}$  are constants.

Employing the chain rule, the temperature gradient is expressed as

$$\frac{\partial \Theta}{\partial n_i} = \frac{\partial \Theta}{\partial U} \frac{\partial U}{\partial n_i} = \frac{k_0}{k(\Theta)} \frac{\partial U}{\partial n_i} \quad (\text{A.8})$$

Introducing Eqs. (5) and (A.8) into (A.7) we obtain

$$k(\Theta) \frac{\partial \Theta}{\partial n_i} + h_{0,i} \int_{\Theta_0}^{\Theta} \frac{k(\Theta')}{k_0} d\Theta' = f_i(\mathbf{r}, t) \quad (\text{A.9})$$

A comparison of Eqs. (A.6) and (A.9) reveals that

$$h_i(\Theta) = h_{0,i} \frac{1}{\Theta} \int_{\Theta_0}^{\Theta} \frac{k(\Theta')}{k_0} d\Theta' \quad (\text{A.10})$$

Thus, when  $h_i(\Theta)$  satisfies the restriction imposed on by (A.10), the nonlinear boundary condition (A.6) is transformed into a linear boundary condition in  $U$  given by Eq. (A.7) under the transformation Eq. (5).